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The Relation of Optima and Market Equilibria with Externalities

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1. INTRODUCTION

Three main theorems provide the basis for welfare economics. These indicate sufficient conditions under which (a) there exists a competitive equilibrium, (b) competitive equilibria are Pareto-optimal, and (c) Pareto-optima are supportable by competitive equilibria. One condition found in all three is that the economic environment be decomposable. That is, the admissible actions and preferences of any agent should be independent of the actions of the other agents. Furthermore, demonstration of the existence of indecomposabilities, usually called externalities, has generally been considered sufficient evidence of "market failure" and of the need for remedies. One such remedy is that the externalities be "internalized." That is, agents who have an impact on each other should get together and make a joint decision. This indicates that there are informational, as well as technological, reasons for market failure due to externalities.

In this paper, the impact of externalities on conclusions (b) and (c), above, is investigated. It is shown that, even if externalities exist, the information structure may be such that (b) and (c) still hold. Delaying definitions until later, we show that (b) holds with externalities in the production sector if information is extensive enough, and (c) holds with externalities in production and consumption if information is limited enough. It is the relation between apparently possible actions and actually possible actions which provides the basis for these results. These concepts are defined in Section 3, and their implications for consumer and producer behavior in a competitive economy are explored in Sections 4 and 5. The rest of the paper is devoted to indicating the impact, on (b) and (c), of externalities. In the next section, a description of the economic environment is presented which includes most of the notations used in the rest of the paper.

2. THE ENVIRONMENT

An economy is considered in which there are n consumers, indexed by $i = 1,..., n$, and m producers, indexed by $j = 1,..., m$. It is assumed that the commodity space C is the Euclidean space and that the actions of any consumer, x^i, and any producer, y^j, belong to C.

The set of *admissible joint* consumptions A is contained in $C^{(n)}$, (the n-fold Cartesian product of the commodity space). The set of *jointly feasible productions* is contained in $C^{(m)}$. Furthermore, if S is the set of *possible states* s of the economy, where $s = (x^1,..., x^n, y^1,..., y^m) \in S$, then $S = X \otimes Y$, ($\otimes$ denotes Cartesian Product). That is, consumption and production decisions are decomposable.

Each consumer has (i) a *complete, reflexive, transitive preordering*, $\succcurlyeq_i$, over the set X, and (ii) an *initial resource vector*, $w^i \in C$.

In summary, the *environment* is:

$i = 1,..., n$: consumers.

$j = 1,..., m$: producers.

C : the commodity space.

$x^i \in C$: the action of consumer i.

$y^j \in C$: the action of producer j.

$S \subsetneq C^{(n+m)}$: the set of possible states, where

$$s = (x^1,..., x^n, y^1,..., y^m) \in S \equiv X \otimes Y.$$

$\succcurlyeq_i$: the preordering of i on X.

$w^i \in C$: the initial resources of i.

3. EXTERNALITIES AND INFORMATION

3(a). *Production*

If $\bar{Y}^j$ is the projection of Y on the j-th component of $C^{(m)}$, then $Y \subsetneq \bar{Y}^1 \otimes \cdots \otimes \bar{Y}^m$. If equality holds, then production is decomposable. This is traditionally what has been meant by a lack of externalities, (e.g., Hurwicz [3]). Negation of equality indicates that externalities are present, although it does not indicate whether there are economies or diseconomies. For our purposes, it is not necessary to distinguish between the two; however, some of the explanatory examples below indicate how these, and other, traditional concepts can be interpreted within the context of this paper. What is important is the relation between what producers think they can do (apparent production possibilities) and what they really can do (actual production possibilities).

The actual possibilities of the production sector are given by the technology Y. This determines the possibilities for aggregate production input-output decisions.

DEFINITION 1. The *net production possibilities* is a set,

$$Y^* = \left\{ z \in C \mid z = \sum_{j=1}^{m} y^j, (y^1,..., y^m) \in Y \right\}.$$

This is related to Y by

LEMMA 1. *If* $Y = \bar{Y}^1 \otimes \cdots \otimes \bar{Y}^m$ *(Y is decomposable), then*

$$Y^* = \sum_{j=1}^{m} \bar{Y}^j.$$

While the converse is not generally true, it is the additivity of production possibilities ($Y^* = \sum_{j=1}^m \bar{Y}^j$) which is used in most welfare theorems involving the competitive mechanism. Thus, the concept of net production possibilities will be used in specifying the relations of apparent and actual possibilities.

Each producer is assumed to possess a correspondence (a set-valued mapping) from Y to C, call it $Y_0^j : Y \to C$, which specifies what he thinks he can do, given the current actions of others. Some examples which will be used later are:

EXAMPLE 1. $Y_0^j(y) = \bar{Y}^j, y \in Y$. This is most often used in papers on traditional welfare economics (see, e.g., Debreu [2]). Each producer considers all those things he can do if the others will only go along with him. Alternatively, he assumes he can force the others to go along with him. For example, the upstream firms on a river, because of property rights and locational advantages, possess these capabilities with respect to downstream firms.

EXAMPLE 2. $Y_0^j(y) = \{ y^j \in C \mid (0,..., 0, y^j, 0,..., 0) \in Y\}$, for all $y \in Y$. That is, each producer thinks the impact of the other firms will be as if they did not produce.

EXAMPLE 3. $Y_0^j(y^*) = \{ y^j \in C \mid (y^{*1},..., y^j,..., y^{*m}) \in Y\}$. That is, each producer considers only those actions which he can undertake without making the proposed actions of others y^* infeasible. One might argue that this involves more information than the first two examples since it requires, if there exist indecomposabilities, knowledge of all of Y as well as y^*.

As indicated earlier, it is not the existence of externalities *per se* which creates problems for the competitive mechanism, but their combination with the informational structure. Two possible relationships are defined.

DEFINITION 2. (a) there exists *extensive production information* at $y^* \in Y$ if and only if $Y^* \subseteq \sum_{j=1}^m Y_0^j(y^*)$.

(b) There exists *limited production information* at $y^* \in Y$ if and only if $\sum_{j=1}^m Y_0^j(y^*) \subseteq Y^*$.

Remark 1. Under Example 1 above there is extensive production information. It should also be noted that both (a) and (b) of Definition 2 represent cases of misinformation. That is, correct information exists at $y^* \in Y$ only if $Y^* = \sum_{j=1}^m Y_0^j(y^*)$.

It is perhaps illuminating to consider two traditional examples of externalities and their relation to the above.

EXAMPLE 4. One usual example of external diseconomies is the case of two firms on a river, where the upstream firm is a polluter and the downstream firm is a user of water. If we assume that $Y_0^u(y^*) = \bar{Y}^u$ for all $y^* \in Y \subseteq \bar{Y}^u \otimes \bar{Y}^d$, and $Y_0^d(y^*) = \{ y^d \in C \mid (0, y^d) \in Y\}$ for all $y^* \in Y$, then, since the actual possibilities for d decrease as u's output increases, there is extensive production information at all $y^* \in Y$. That is, producers think they can do more than they actually can.

EXAMPLE 5. In the case of external economies for two firms, it is assumed, e.g., that $Y_0^1(y) = \bar{Y}^1$ for $y \in Y$, and $Y_0^2(y) = \{ y^2 \in C \mid (0, y^2) \in Y\}$. Also, if $\bar{Y}^2(y^1) = \{ y^2 \in C \mid (y^1, y^2) \in Y\}$, then $\bar{Y}^2(y^1) \supseteq \bar{Y}^2(0)$ for all $y^1 \in \bar{Y}^1$. In this case, there is limited production information for all $y \in Y$. That is, there exist unknown possibilities.

These two examples indicate that, to a limited extent, Definition 2 corresponds to traditional views of externalities. However, this correspondence obviously rests on the particular form of $Y_0^j(y)$ and Y. In effect it is a problem with both technological and information dimensions.

3(b). *Consumption*

Consumer information involves both admissibility X and preferences $\succcurlyeq_i$. For the purposes of this paper, it is appropriate that both be handled simultaneously.

DEFINITION 3. (a) The *net admissible consumption* set is

$$X^* = \left\{ z \in C \mid z = \sum_{i=1}^{n} x^i, (x^1,..., x^n) \in X \right\}.$$

(b) The *net preferred consumption* set of $\bar{x} \in X$ is

$$\hat{X}^*(\bar{x}) = \left\{ z \in X^* \mid z = \sum_{i=1}^{N} x^i, x \succcurlyeq_i \bar{x} \quad \text{for} \quad i = 1,..., n \right\}.$$

If $X = \bar{X}^1 \otimes \cdots \otimes \bar{X}^n$ and if preferences are selfish (i.e., depend only on x^i), letting $\hat{X}^i(\bar{x}^i) = \{x^i \in \bar{X}^i \mid x^i \succcurlyeq_i \bar{x}^i\}$, we have $X^* = \sum_{i=1}^{N} \bar{X}^i$ and $\hat{X}^*(\bar{x}) = \sum_{i=1}^{N} \hat{X}^i(\bar{x}^i)$.

Each consumer is assumed to possess a correspondence $X_0^i : X \to C$, which specifies what he thinks he can do. Each consumer is also assumed to possess an apparent preference mapping $\hat{X}_0^i : X \to C$, where $\hat{X}_0^i(\bar{x})$ is specified as

$$\hat{X}_0^i(\bar{x}) = \{x^i \in X_0^i(\bar{x}) | (\bar{x}^1,..., x^i,..., \bar{x}^n) \succcurlyeq_i \bar{x}\}.$$

That is, $\hat{X}_0^i(\bar{x})$ is the set of actions i thinks he can unilaterally undertake which would make him at least as well-off.

DEFINITION 4. (a) There exists *extensive consumer information* at $x^* \in X$ if and only if $\hat{X}^*(x^*) \subseteq \sum_{i=1}^{n} \hat{X}_0^i(x^*)$.

(b) There exists *limited consumer information* at $x^* \in X$ if and only if $\sum_{i=1}^{n} \hat{X}_0^i(x^*) \subseteq \hat{X}^*(x^*)$.

Since so many examples were provided for the production section, we leave, with one exception, this task for consumption to the reader. We consider the case of a pure public commodity (see Samuelson [5]).

EXAMPLE 6. Assume there are K commodities, with K-th being a public good. We assume that

$X \equiv \{x \in \bar{X}^1 \otimes \cdots \otimes \bar{X}^n \mid x_K^1 = x_K^n\}$, and that preferences are selfish.

(a) If for each i,

$$X_0^i(\bar{x}) = \{x^i \in \bar{X}^i | (\bar{x}^1,..., x^i,..., \bar{x}^n) \in X\},$$

then there is limited consumption information. We have, for all i, $\hat{x}^i \in \hat{X}_0^i(\bar{x})$ implies $\hat{x}^i \succcurlyeq \bar{x}^i$ and $\hat{x}^i \in \bar{X}^i \cap \{x^i \in \bar{X}^i \mid x_K^i = \bar{x}_K^i\}$. Hence, $\sum_{i=1}^{n} \hat{x}^i \in X^*$ and $\hat{x}^i \succcurlyeq \bar{x}^i$ for all i, or $\sum_{i=1}^{n} \hat{x}^i \in \hat{X}^*(\bar{x})$.

(b) If $X_0^i(\bar{x}) = \bar{X}^i$, then there is extensive consumption information. The proof is straightforward.

This example serves to indicate that, even with the same technological structure, information may be either limited or extensive.

4. EXTERNALITIES, INFORMATION AND PROFIT MAXIMIZATION

In this section, the impact of information about externalities on profit maximization is considered. Two lemmas are presented which summarize the relationship between individual and joint profit maximization.

ASSUMPTION 1. Given $\hat{y} \in Y$, and $p \in C$, producer j will choose $y^{*j} \in Y^j_{0(\hat{y})}$, such that

$$py^{*j} = \max p\, Y_0^j(\hat{y}),$$

where

$$pY_0^j(\hat{y}) = \{r \in (-\infty, +\infty) | \, r = p \cdot z, z \in Y_0^j(\hat{y})\}.$$

Even if there is extensive production information, individual profit maximization at a possible production implies joint profit maximization. That is,

LEMMA 2. *If $\hat{y} \in Y$, $Y^* \subseteq \sum_{j=1}^{m} Y_0^j(\hat{y})$ and $p\hat{y}^j = \max p\, Y_0^j(\hat{y})$ for $j = 1,..., m$, then $p \cdot \sum_{j=1}^{m} \hat{y}^j = \max p \cdot Y^*$.*

Proof. Assume there exists $\hat{z} \in Y^*$ such that $p \cdot \hat{z} > p \cdot \sum_{j=1}^{m} \hat{y}^j$. $\hat{z} \in Y^* \subseteq \sum_{j=1}^{m} Y_0^j(\hat{y})$ implies there exist, for each j, $\bar{y}^j \in Y_0^j(\hat{y})$ such that $\hat{z} = \sum_{j=1}^{m} \bar{y}^j$. Also, $p \cdot \hat{z} > p \sum_{j=1}^{m} \hat{y}^j$ implies there is at least one k such that $p \cdot \bar{y}^k > p \cdot \hat{y}^k$. But $p\hat{y}^k \geqslant p\bar{y}^k$, since $p\hat{y}^k = \max p \cdot Y_0^k(\hat{y})$. Q.E.D.

Remark 3. If $\hat{y} \notin Y$, the conclusion of the lemma need not hold. Although this is unimportant for this paper, since we are only worried about implementable equilibrium productions (see the next section), it points out a potential advantage for adjustment processes which remain feasible (e.g., some nontatonnement processes) over those which do not (e.g., a tatonnement process like the usual interpretation of the competitive mechanism).

The statement which is almost the converse of Lemma 2 is that, even if there is limited production information, joint profit maximization implies individual profit maximization.

LEMMA 3. *If $\sum_{j=1}^{m} Y_0^j(\hat{y}) \subseteq Y^*$, $\hat{y}^j \in Y_0^j(\hat{y})$ $\forall$ $j = 1,..., m$ and $p \sum_{j=1}^{m} \hat{y}^j = \max pY^*$, then $p\hat{y}^j = \max pY_0^j(\hat{y})$ for all $j = 1,..., m$.*

Proof. Assume there is a k and $\bar{y}^k$ such that $\bar{y}^k \in Y_0^j(\hat{y})$ and $p \cdot \bar{y}^k > p \cdot \hat{y}^k$. Let $\bar{y}^j = \hat{y}^j$ for all $j \neq k$. Since $\bar{y}^j \in Y_0^j(\hat{y})$ for all j, then $\bar{z} = \sum_{j=1}^m \bar{y}^j \in Y^*$. But $p\bar{x} = p\bar{y}^k + \sum_{j=1,j=k}^m p\hat{y}^j > p \cdot \sum_{j=1}^m \hat{y}^j$. Hence $p \sum_{j=1}^m \hat{y}^j \neq \max p \cdot Y^*$. Q.E.D.

5. Externalities, Information, Utility Maximization and Expenditure Minimization

The impact of information about externalities on consumer behavior in a competitive adjustment mechanism is explored in this section.

Assumption 2. Given $\hat{x} \in X$, a wealth scalar v^i, and $p \in C$, consumer i will choose $x^{*i} \in X_0^i(\hat{x})$ such that $px^{*i} \leqslant v^i$ and

$$(\hat{x}^1, \ldots, x^{*i}, \ldots, \hat{x}^n) \succcurlyeq_i (\hat{x}^1, \ldots, x^i, \ldots, \hat{x}^n)$$

for all $x^i \in X_0^i(\hat{x})$ such that $px^i \leqslant v^i$. This corresponds to the idea of the best replay used in game theoretic models. That is, i chooses the most preferred action he thinks he can unilaterally implement, given the actions of all other consumers. In a private ownership economy, v^i will be the sum of the profit shares received by i and the value of w^i, his initial resource holdings.

Lemmas similar to those in Section 4 apply to the relation between joint and individual consumer behavior. Even if there is extensive consumption information, individual expenditure minimization implies joint expenditure minimization.

Lemma 4. *If $x^* \in X, \hat{X}^*(x^*) \subseteq \sum_{i=1}^n \hat{X}_0^i(x^*)$, and $px^{*i} = \min p\hat{X}_0^i(x^*)$ then $p \sum x^{*i} = \min p\hat{X}^*(x^*)$.*

Proof. This proof follows in the same way that the one for Lemma 2 does. Q.E.D.

Even if there is limited consumer information, joint expenditure minimization implies individual expenditure minimization.

Lemma 5. *If $\sum_{i=1}^n \hat{X}_0^i(\bar{x}) \subseteq \hat{X}^*(\bar{x})$, $\bar{x}^i \in \hat{X}_0^i(\bar{x})$ $\forall$ $i = 1, \ldots, n$ and if $p \sum \bar{x} = \min p\hat{X}^*(\bar{x})$, then $p\bar{x}^i = \min p\hat{X}_0^i(\bar{x})$ for all i.*

Proof. Like that for Lemma 3.

The relationship between Pareto-optimal states and competitive equilibria (discussed in Section 7) depends on the relationship between utility maximization (Assumption 2) and expenditure minimization.

Lemma 5. *If $\succcurlyeq_i$ satisfies local nonsatiation (i.e., given $x^* \in X$, $\delta > 0$, there exists $z^* \in X$ such that $z^{*k} = x^{*k}$ for all $k \neq i$, $\| z^* - x^* \| < \delta$, and $z^* \succ_i x^*$), then* (a) $p\bar{x}^i = \min p\hat{X}_0^i(\bar{x})$ *if* (b) $p\bar{x}^i \leqslant v^i$, $(\bar{x}^1, \ldots, x^i, \ldots, \bar{x}^n) \succ_i \bar{x}$, *and $x^i \in X_0^i(\bar{x})$ imply $px^i > v^i$.* [(b) *is equivalent to $\bar{x}^i$, having been chosen under Assumption* 2, *given $\bar{x}$.*]

Proof. Assume that (a) does not hold. Then there exists $x^{*i} \in \hat{X}_0^i(\bar{x})$ such that $px^{*i} < v^i$. By local nonsatiation, there exists $z^i \in X_0^i(\bar{x})$ such that $p \cdot z^i \leqslant v^i$ and $(\bar{x}^1, \ldots, z^i, \ldots, \bar{x}^n) \succ_i (\bar{x}^1, \ldots, x^{*i}, \ldots, \bar{x}^n) \succcurlyeq_i \bar{x}$. Hence (b) is violated. Q.E.D.

Lemma 6. *If $X_0^i(x)$ is convex, $\{x^i \in X_0^i(\bar{x}) | \, \bar{x} \succcurlyeq_i (x^2, \ldots, x^i, \ldots, \bar{x}^n)\}$ is closed, and $p\bar{x}^i \neq \min pX_0^i(\bar{x})$, then* (a) implies (b).

Proof. Almost identical to that in Debreu [3, p. 69(1)]. Q.E.D

After definitions of Pareto-optimality and competitive equilibrium are presented in Section 5, the implications of Lemmas 2 and 6 are explored in section 7.

6. Pareto-optimality and Competitive Equilibrium; Definition

A state s of an economy E is Pareto-optimal for E if it is attainable (to be defined) and if no one can be made better off, with respect to $\succcurlyeq_i$, without making someone worse off.

Definition 4. Given an economy

$$E \equiv \{i = 1, \ldots, n, j = 1, \ldots, m, S, \succcurlyeq_1, \ldots, \succcurlyeq_n, w^1, \ldots, w^n\},$$

a state $s = (x, y)$ is *attainable* if $s \in A_E$ where

$$A_E \equiv \left\{ s^* \in S \,\middle|\, \sum_{i=1}^n x^{*i} - \sum_{j=1}^m y^{*j} = \sum_{i=1}^n w^i \right\}.$$

Definition 5. Given an economy E, a state $s = (x, y)$ is Pareto-optimal for E if

(a) $s \in A_E$, and

(b) there exists no $s^* \in A_E$,

such that $x^* \succcurlyeq_i x$ for all $i = 1, \ldots, n$, and $x^* \succ_k x$ for some $k = 1, \ldots, n$.

A competitive equilibrium is the equilibrium state of some adjustment process, which will remain unspecified. A distinction is made between implementable equilibria and others. If $S = X \circ Y$ is decomposable, as is usually assumed, the distinction need not be made. However, the introduction of externalities requires it (see Lemma 2). An implementable competitive equilibrium, given $p \in C$, is an attainable state such that individual producers apparently maximizing profits and individual consumers are apparently maximizing utility (representing $\succcurlyeq$) subject to a wealth constraint determined by the initial resources and the share of the profits of producers.

DEFINITION 6. Given $p \in C$ and $s^* \in S$, the *wealth* of consumer i is $v^i(p, s^*) = p \cdot w^i + \sum_{j=1}^m \theta_{ij} p \cdot y^{*j}$ where θ_{ij} is the fraction of profits of producer j received by consumer i. Hence, $0 \leqslant \theta_{ij} \leqslant 1$, for all i and j, and $\sum_{i=1}^n \theta_{ij} = 1$ for all $j = 1,..., m$.

DEFINITION 7. $(s^*, p^*) \in C^{n+m+1}$ is an implementable competitive equilibrium of an economy E if

(1) $s^* \in A_E$,

(2) $p^* y^{*j} = \max p^* Y_0^j(y^*)$ for all $j = 1,..., m$, and

(3) for all $i = 1,..., n$,

(a) $p^* \cdot x^{*i} \leqslant v^i(p^*, s^*)$ and

(b) $x^* \succcurlyeq_i (x^{*1},..., x^i,..., x^{*n})$ for all $x^i \in \{x^i \in \hat{X}_0^i(x^*) \mid p^* \cdot x^i \leqslant v^i(p^*, s^*)\}$.

7. Pareto-Optima and Competitive Equilibria: Their Relationship

Some welfare economics of competitive equilibrium are explored in this section. The first result, embodied in Theorem 1, is that, *even with extensive production information*, an implementable competitive equilibrium is Pareto-optimal if consumer information is exact, and preferences $\succcurlyeq_i$ satisfy local nonsatiation. This result obviously extends the environmental coverage usually considered in this type of theorem (see, e.g., Arrow [1], Debreu [2], and Koopmans [4]). The second result, embodied in Theorem 2, is that *even with limited production or consumption information*, there exist prices and a reallocation of initial resources and profit shares such that a Pareto-optimum can be supported by a competitive equilibrium under four other conditions. These are continuity, and regular convexity of consumer preferences, and the convexity of X and Y. This theorem also contains an extension of the usual environmental coverage. (see op. cit.)

THEOREM 1. *Given an economy* $E = \{i, j, S, \succcurlyeq_i, w^i, \theta_{ij}\}$, *where* $i = 1,..., n, j = 1,..., m$, *such that*

(a) $S = \bar{X}^1 \otimes \cdots \otimes \bar{X}^n \otimes Y$, *and*

(b) $\succcurlyeq_i$ *satisfies local nonsatiation (see Lemma 5) for* $i = 1,..., n$.

If (s^*, p) *is an implementable competitive equilibrium of E such that*

(c.1) $Y^* \subseteq \sum_{j=1}^n Y_0^j(y^*)$, *and*

(c.2) $\hat{X}(x^*) = \sum_{i=1}^n \hat{X}_0^i(x^*)$,

then s^* *is Pareto-optimal.*

Proof. By local nonsatiety, (c.2), and Lemma 5, we get $px^{*i} = \min p\hat{X}_0^i(x^*)$ for all i. Hence, by Lemma 4, $p \sum_{i=1}^n x^{*i} = \min pX^*(x^*)$.

By (c.1) and Lemma 2, $p \sum y^{*i} = \max pY^*$. Therefore,

$$p\left(\sum_i w^i\right) = p\left(\sum x^{*i} - \sum y^{*j}\right) = \min p[X^*(x^*) - Y^*].$$

Let $\hat{s} \succcurlyeq_i s^*$ for all i, where $\hat{s} \in A_E$. Then $\sum_{i=1}^n \hat{x}^i - \sum_{j=1}^m \hat{y}^j = \sum w^i$. Therefore, $p \sum_{i=1}^n \hat{x}^i - p \sum_{j=1}^m \hat{y}^j = \min p[X^*(x^*) - Y^*]$. But

$$X^*(x^*) = \sum_{i=1}^n \hat{X}_0^i(x^*)$$

and $\hat{x}^i \in \hat{X}_0^i(x^*)$ for each i imply, by Lemma 5, that

$$p\hat{x}^i = \min p\hat{X}_0^i(x^*) = px^{*i}$$

for every i. Hence, $\hat{x} \underset{i}{\sim} x^*$ for all i and x^* is Pareto-optimal. Q.E.D.

THEOREM 2. *Given an economy* $E = \{i, j, S, \succcurlyeq_i, w^i, \theta_{ij}\}$ *such that*

(a) $S = X \otimes Y \subseteq [(n + m)\ K$ *dimensional Euclidean space*],

(b.1) X *is convex*,

(b.2) *for every* $x' \in X$, *the sets* $\{x \in X \mid x \succcurlyeq_i x'\}$ *and* $\{x \in X \mid x' \succcurlyeq_i x\}$ *are closed in X for each i.*

(b.3) *if* $\bar{x}, x' \in X$ *and* $\lambda \in (0, 1)$. *then* $\bar{x} \succ_i x'$ *implies* $\lambda\bar{x} + (1 - \lambda)\, x' \succ_i x'$, *for each i, and*

(c) Y *is convex.*

If s^* *is Pareto-optimal for E*,

(d.1) $\sum_{i=1}^n \hat{X}_0^i(x^*) \subseteq \hat{X}(x^*)$, $x^{*i} \in \hat{X}_0^i(x^*)$,

(d.2) $\sum_{j=1}^m Y_0^j(y^*) \subseteq Y^*$, $y^{*i} \in Y_0^i(y^*)$, *and* (d.3) *there is some* k' *and some* $x^{k'} \in \hat{X}_0^{k'}(x^*)$ *such that* $(x^{*1}, \ldots, x^{k'}, \ldots, x^{*n}) \succ_{k'} x^*$, *then there exists* $p \in C$, $p \neq 0$, *such that*

(α) $p \cdot y^{*j} = \max pY_0^j(y^*)$ *for all j, and*

(β) $p \cdot x^{*i} = \min p\hat{X}_0^i(x^*)$ *for all i.*

Proof (Similar to Debreu [2], p. 96). Let $\hat{X}^0(x^*) = \{z \in C \mid z = \sum_{i=1}^n x^i$, where $x \succcurlyeq_i x^*$ for all i, $x \succ_{k'} x^*$ for $k'\}$. Then, letting $\mathring{H} = \hat{X}^0(x^*) - Y^*$, by (b.1), (b.3), and (c), $\mathring{H}$ is convex. By (d.3), $\mathring{H} \neq \varnothing$, and, since s^* is optimal, $w = \sum_{i=1}^n w^i \notin \mathring{H}$. Hence, by Minkowski's separation theorem, there exists $p \neq 0$ such that $p \cdot z \geqslant p \cdot w$ for all $z \in \mathring{H}$. By (b.3) and (d.3), if $\bar{x} \sim_{k'} x^*$, then $\bar{x}$ belongs to the closure of $\{x \in X \mid x \succcurlyeq_{k'} X^*\}$. Therefore, $\hat{X}(x^*) \subseteq \hat{X}$, the closure of $\hat{X}^0(x^*)$, and $H = \hat{X}(x^*) - Y^*$ is contained in the closure of $\mathring{H}$. Therefore, $H \subseteq \{z \in x \mid p \cdot z \geqslant p \cdot w\}$. But $w \in H$ since $s^* \in H$ and $p \cdot w = \min pH = \min p[\hat{X}(x^*) - Y^*]$. Therefore, by (d.1), (d.2) and Lemmas 3 and 5, $px^{*i} = \min p\hat{X}_0^i(x^*)$ for all i and $-py^{*j} = \min p\hat{Y}_0^j(y^*)$ for all j. Q.E.D.

COROLLARY T.2.1. *If, in addition,* $px^{*i} \neq \min pX_0^i(x^*)$ *for any* $i = 1, \ldots, n$, *then there exist* $\bar{w}^1, \ldots, \bar{w}^n, \bar{\theta}_{11}, \ldots, \bar{\theta}_{nm}$ *such that* (s^*, p) *is an implementable competitive equilibrium for E.*

Proof. If $px^{*i} \neq \min X_0^i(x^*)$ for any i, then, by Lemma 6, $px^{*i} = \min p\hat{X}_0^i(x^*)$ implies that, for any $\bar{v}^i = px^{*i}$, (x^{*i}, p) satisfies (3) of Definition 7. Letting $\bar{w}^i = x^{*i} - (1/n)\sum_{j=1}^m y^{*j}$ and $\bar{\theta}_{ij} = (1/n)$, we have $\bar{v}^i = p[x^{*i} - (1/n)\sum_{j=1}^m y^{*j}] + \sum_j (1/n)\, py^{*j} = px^{*i}$. Q.E.D.

8. ADDITIONAL REMARKS

An important relationship in the theorems should be noticed. The assumptions needed in Theorems 1 and 2 depend on the definition of competitive equilibrium since this implicitly determines the information correspondence requirements relative to the economic environment. Two examples come to mind. If Definition 7(3) were revised to read either

DEFINITION 7(3′). For each i, $x \succ_i x^*$ implies $px^i > v^i$, or

DEFINITION 7(3″). For any $i = 1, \ldots, n$, $x \succ_i x^*$ implies $p\sum_{i=1}^n x^i > \sum_{i=1}^n v^i$,

then, in Theorem 1, assumption (a) could be weakened to allow for extensive consumer information. That is, if $\hat{X}(x^*) \subseteq \sum_{i=1}^n \hat{X}_0^i(x^*)$, Theorem 1 would still be applicable.

Although it is difficult to interpret 7(3′), game theorists will recognize that 7(3) corresponds to the noncooperative (Nash) solution of a $(n + m)$ person game, while 7(3″) corresponds to a cooperative solution of the same game. Which one should be the appropriate interpretation of the competitive process? I leave that decision to the reader.

Although Theorems 1 and 2 extend the traditional welfare theorems when treated individually, it is true that if both are to hold simultaneously, then consumption and production information must be exact. That is, $\hat{X}(x^*) = \sum_{i=1}^n \hat{X}_0^i(x^*)$ and $Y^* = \sum_{j=1}^m Y_0^j(y^*)$. Thus, if $Y_0^j(y^*) = \bar{Y}^j$ for all $y^* \in Y$ and $\hat{X}_0^i(x^*) = \{x^i \in \bar{X}^i \mid x^i \succcurlyeq_i x^{*i}, \succcurlyeq_i$ is selfish$\}$, then we still require no externalities if both theorems are to hold under the same set of assumption. However, if the information correspondences are different, then, even if externalities exist, competitive equilibria can be Pareto-optima and vice versa.

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